Tutorial Note VII

In this notes, we present other methods to derive the solution formula for inhomogeneous 1D wave equations:

$$\begin{cases} u_{tt} - u_{xx} = f; \\ u(0,x) = 0, \quad u_t(0,x) = 0. \end{cases}$$

1 Method of Characteristic Coordinates

Consider the characteristic coordinates:

$$\begin{cases} \xi = x + t; \\ \eta = x - t. \end{cases}$$

Then

$$\begin{cases} \partial_x = \partial_\xi + \partial_\eta; \\ \partial_t = \partial_\xi - \partial_\eta. \end{cases}$$

So in the characteristic coordinates, the wave equations turn to be

$$-4u_{\xi\eta} = f.$$

The conditions on initial data actually give us that

$$u(0,x) = 0, \quad \nabla u(0,x) = 0.$$

Since in the characteristic coordinates, the line $t = 0$ turns to be $\xi = \eta$, we have

$$u(s,s) = 0, \quad \nabla u(s,s) = 0.$$

For a point (ξ_0, η_0), we have

$$\begin{aligned} u(\xi_0, \eta_0) &= \int_{\eta_0}^{\xi_0} u_\xi(p, \eta_0)\,\mathrm{d}p \\ &= \int_{\eta_0}^{\xi_0} \int_p^{\eta_0} u_{\xi\eta}(p,q)\,\mathrm{d}q\,\mathrm{d}p \\ &= \int_{\widetilde{\triangle}} \frac{f}{4}, \end{aligned}$$

where $\widetilde{\triangle}$ denotes the triangle with vertices (η_0, η_0), (ξ_0, ξ_0), (ξ_0, η_0). Next we show that

$$\int_{\widetilde{\triangle}} \frac{f}{4} \,\mathrm{d}\xi \,\mathrm{d}\eta = \int_{\triangle} \frac{f}{2} \,\mathrm{d}t \,\mathrm{d}x, \tag{1}$$

where $\triangle$ denotes the characteristic triangle in the original coordinate with vertices (t_0, x_0), $(0, x_0 - t_0)$, $(0, x_0 + t_0)$, where (t_0, x_0) is the original coordinate of (ξ_0, η_0). Now it is easy to see that $\triangle$ and $\widetilde{\triangle}$ are the same triangle in the two coordinates. The Jacobian of the coordinate transformation is

$$\begin{pmatrix} \frac{\partial \xi}{\partial t} & \frac{\partial \xi}{\partial x} \\ \frac{\partial \eta}{\partial t} & \frac{\partial \eta}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

So (1) follows from the formula of change of variables.

2 Method of Green's Functions

The fundamental solution of 1D wave equations is $1_{|x|<t}/2$. So we could use the fundamental solution to find the solution formula: $f * 1_{|x|<t}/2$, which is

$$\frac{1}{2} \int_{\triangle} f,$$

where $\triangle$ denotes the characteristic triangle. Next we verify it.

$$\begin{aligned} \frac{1}{2} \int_{\triangle} f &= \frac{1}{2} \int_{\triangle} (u_{tt} - u_{xx}) \\ &= \frac{1}{2} \int_{\partial\triangle} (-u_x, u_t) \cdot \vec{n}. \end{aligned}$$

$\partial\triangle$ consists of three lines: $L_1 = [(x_0 - t_0, 0), (x_0 + t_0, 0)]$,$L_2 = [(x_0 + t_0, 0), (x_0, t_0)]$, $L_3 = [(x_0, t_0), (x_0 - t_0, 0)]$. Since the initial data vanish, the integral on L_1 is 0. Choose a parametrization of L_2: $(x_0 + t_0 - s, s)$, $0 \le s \le t_0$. Then

$$\begin{aligned} &\frac{1}{\sqrt{2}} \int_{L_2} (-u_x + u_t) \,\mathrm{d}l \\ &= \int_0^{t_0} (-u_x(x_0 + t_0 - s) + u_t(x_0 + t_0 - s, s)) \,\mathrm{d}s \\ &= \int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}s} u(x_0 + t_0 - s, s) \,\mathrm{d}s \\ &= u(x_0, t_0). \end{aligned}$$

Similarly, choose a parametrization of L_3: $(x_0 - t_0 + s, s)$, $0 \le s \le t_0$. Then

$$\begin{aligned} &\frac{1}{\sqrt{2}} \int_{L_3} (u_x + u_t) \,\mathrm{d}l \\ &= \int_0^{t_0} (u_x(x_0 - t_0 + s, s) + u_t(x_0 - t_0 + s, s)) \,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{t_0} \frac{\mathrm{d}}{\mathrm{d}s} u(x+0-t_0+s, s)\,\mathrm{d}s \\
&= u(x_0, t_0).
\end{aligned}
$$

So we have verified it.

3 MVP and Strong Maximum Principle for Heat Equations

Heat equations could be regarded as evolutionary Laplace equations, so properties of Laplace equation often have corresponding versions for heat equations, although usually the versions for heat equations are more complicated because a solution to a Laplace equation could be regarded as a (steady) solution to a heat equation. Here we introduce a MVP for heat equations. We call

$$
H(x,t;\mu) = \{(y,s) \mid K(x,t;y,s) \geq \mu\}
$$

a heat ball.

Theorem 3.1 *Suppose that $u \in C^{2,1}(\Omega_T)$ is a solution to the heat equation and $H(s,t;\mu) \subset \Omega_T$. Then*

$$
u(x,t) = \mu \int_{H(x,t;\mu)} u(y,s) \left(\frac{|x-y|}{2(t-s)} \right)^2 \mathrm{d}y\,\mathrm{d}s.
$$

Here $C^{2,1}(\Omega_T)$ denotes the functions satisfying that $u, \partial_x u, \partial_{xx} u, \partial_t u$ exist and are all continuous in Ω_T. The proof can be found in Evans' PDE section 2.3.2 or Jost's PDE section 5.1.

Similar to Laplace equations, with a MVP, we could obtain the strong maximum principle.

Theorem 3.2 *Suppose that $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ is a solution to the heat equation. If u attains its maximum in Ω_T, then u is constant.*

The proof is similar to Laplace equations and can be found in Evans' PDE section 2.3.3.